TRANSFORMATIONS OF THE FUNDAMENTAL GROUPS CORRESPONDING TO THOSE OF HEEGAARD DIAGRAMS BY THE BAND MOVES

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1. Introduction

H. Zieschang obtains an important result concerning the band move for a handlebody (§3 Theorem 4). We will give transformations by the band move for basic Heegaard diagrams (§3 Theorem 5). From the Theorems 4 and 5, we get more developed transformations for Heegaard diagrams (§3 Theorem 6). F. Waldhausen obtains an important result concerning the equivalent of Heegaard splittings for the 3-sphere S^3 (§3 Theorem 7). Moreover, from the Theorems 6 and 7, we get a distinguishing feature of transformations of Heegaard diagrams for S^3 (§3 Theorem 8).

§4 deals with the main theme that the transformation by a band move of the Heegaard diagram and that of the fundamental group by an algebra calculation (replacements or substitution) correspond to 1 to 1 (Theorem 9). Moreover, a reduction of the fundamental groups for S^3 is obtained (Theorem 10).

Everything in this paper, we will be considering the piecewise linear point of view. ∂X, $Int(X)$, $Cl(X)$ indicates the boundary, interior, closure of a point set X, respectively. Hereafter, notation M^3 denotes a closed, connected orientable 3-manifold unless otherwise stated.

2. Preliminaries

In this section we give definitions, basic Theorems 1 and 2, and so forth.

We begin with a definition of a handlebody.

Definition 1. Let $\{D_1, \cdots, D_n\}$ be mutually disjointed 2-disks and $h_i = D_i \times [0, 1]$ ($i = 1, \cdots, n$). *A handlebody H of genus n* is a 3-ball (cube) B^3 with n handles $\{h_i\}$ so that the result of attaching h_i with homeomorphisms throws $2n$ disks $D_i \times 0$, $D_i \times 1$ onto $2n$ disjointed 2-disks on ∂B^3. H is represented as $B^3 + \{\bigcup_{i=1}^{n} h_i\}$ where $B^3 \cap h_i = \partial B^3 \cap \partial h_i = \{D_i \times 0,\ D_i \times 1\}$. A handlebody H of genus n is also called as *a solid torus of genus n*.

We note that ∂H is an orientable or nonorientable closed surface of Euler characteristic

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$2-2n$ according as H is orientable or nonorientable.

Definition 2. Let H be a genus n handlebody and $\{D_i\}$ $(i = 1, \cdots, n)$, mutually disjointed properly embedded 2-disks in H. If the $Cl(H - \{D_1 \cup \cdots \cup D_n\})$ becomes 3-ball, then the collection $\{D_i\}$ $(i = 1, \cdots, n)$ is called *a complete system of meridian disks of* H and $\{\partial D_i\}$ *a complete system of meridian circles of* ∂H.

Note that $\{D_1, \cdots, D_n\}$ cuts ∂H into 2-sphere with $2n$ holes.

Definition 3. Let H be an orientable genus $n(\geqq 2)$ handlebody with the same presentation as in Def. 1.

(1) Fig. 2–1 shows two handles h_i and h_j of H. By an ambient isotopy of H, keeping $D_i \times 0$ fixed, and sliding $D_i \times 1$ along the direction of the line in $\partial(B^3 + h_j)$, h_i goes over the h_j and turns back to the first place. This operation is called *a handle sliding* of h_i about h_j.

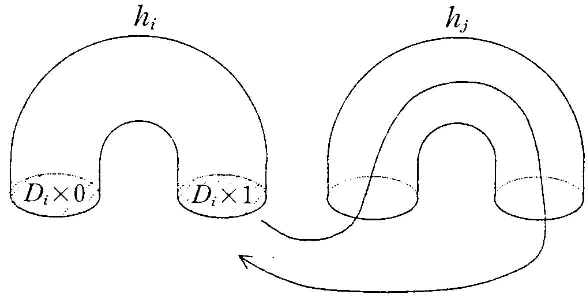


Fig. 2–1

(2) Let $\{D_i\}$ $(i = 1, \cdots, n)$ be a complete system of meridian disks of H and $\{m_i\}$ $(m_i = \partial D_i)$ a complete system of meridian circles of ∂H. Let α be an arc on ∂H that joins two chosen meridians m_i, m_j and $Int(\alpha) \cap \{m_i \cup m_j\} = \phi$. See Fig. 2–2. Let $N(m_i \cup \alpha \cup m_j, \partial H)$ be a regular neighborhood of $m_i \cup \alpha \cup m_j$ on ∂H. ∂N consist of three circles. Out of the three circles, two are isotopic to m_i, m_j and then the remainder is not isotopic to them. Let the notation of remainder be m_{ij}. m_{ij} is called *a band sum of* m_i *and* m_j *(with respect to the band* α*)*. It has also the very pleasant property that bounds a disk and it is homeomorphic to D_i and D_j. Changing the label m_{ij} into m_i (m_j resp.) is called *a band move of* m_i (m_j resp.).

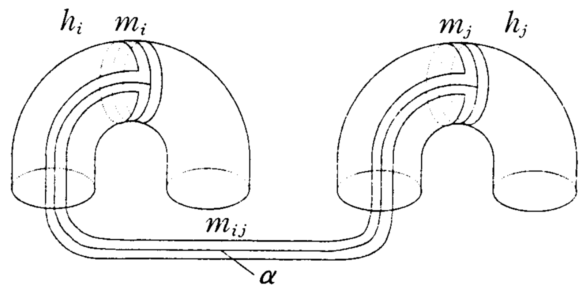


Fig. 2–2

(3) Let $\tilde{D}_i$, $\tilde{D}_j$ be a disk in the foot of ∂h_i, ∂h_j shown in Fig. 2–3, respectively. Gluing together h_i and h_j by an orientation-reversing homeomorphism f: $\tilde{D}_i \to \tilde{D}_j$, a handlebody with the deformed part, the figure 3-shape turned to $\pi/4$ radians is obtained (see §3 U1–C). This operation is called as *handles combining with* h_i *and* h_j.

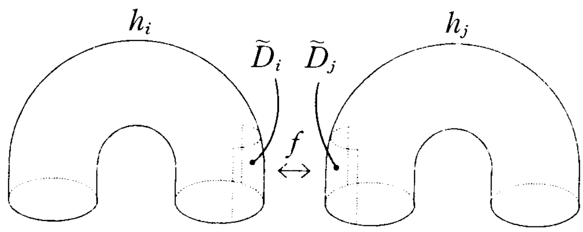


Fig. 2–3

Definition 4. A closed, connected 3-manifold M^3 is represented with a union of two handlebodies H_1, H_2 in M^3; $M^3 = H_1 \cup H_2$ so that $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2$. ∂H_1 $(= \partial H_2)$ is a closed surface of genus $n(\geqq 1)$. Let the surface be F. H_1 (H_2 resp.) and F are orientable or nonorientable according as M^3 is orientable or nonorientable. A triplet (H_1, H_2, F) or $M^3 = H_1 \cup H_2$ is called *a Heegaard splitting* (*H-splitting*) *of* M^3 *with genus* n and H_1 (H_2 resp.), *a Heegaard-handlebody* (*H-handlebody*). F is called *a Heegaard-surface* (*H-surface*) and the integer $n(\geqq 1)$, *Heegaard genus* (*H-genus*). Let U and V be disjointed handlebodies with the same genus. Let $f: U \to V$ be a homeomorphism so that $f|\partial U : \partial U \to \partial V$ is an orientation-reversing homeomorphism. Gluing together ∂U of U and ∂V of V by f, we get M^3. Then M^3 is denoted as $(M^3; U, V, f)$ or $M^3 = U \underset{f}{\cup} V$. It is called *a genus* n *H-splitting of* M^3 *concerning* f.

In $(M^3; U, V, f)$, by replacing $f^{-1}(V)$ with V, one can regard $(M^3; U, V, f)$ as (U, V, F) of M^3.

Theorem 1. *Let* $M^3 = H_1 \underset{\phi}{\cup} H_2$ *and* $M'^3 = H_1' \underset{\phi'}{\cup} H_2'$ *be two H-splittings with the same genus. Suppose that there exist homeomorphisms* $f: H_1 \to H_1'$ *and* $g: H_2 \to H_2'$ *so that the right side diagram becomes commutative. Then* M^3 *is homeomorphic to* M'^3.

$$\begin{tikzcd} \partial H_1 \arrow[r, "\phi"] \arrow[d, "f|\partial H_1"'] & \partial H_2 \arrow[d, "g|\partial H_2"] \\ \partial H_1' \arrow[r, "\phi'"'] & \partial H_2' \end{tikzcd}$$

Proof. Suppose that $h: M^3 \to M'^3$ is a homeomorphism so that $h|\partial H_1 = f$ and $h|\partial H_2 = g$. Then by the above commutative diagram, h is well-defined. □

Theorem 2. *Let* $M^3 = H_1 \underset{\phi}{\cup} H_2$ *be a genus* n *H-splitting and* $\psi: H_1 \to H_1$ *a homeomorphism. Let* $M'^3 = H_1 \underset{\phi\circ(\psi|\partial H_1)}{\bigcup} H_2$. *Then* M'^3 *is homeomorphic to* M^3.

Proof. Let the elliptical character *id.* be an identification map of ∂H_2. Then the right side diagram becomes commutative. Hence by the Theorem 1, M'^3 is homeomorphic to M^3. □

$$\begin{tikzcd} \partial H_1 \arrow[r, "\phi"] \arrow[d, "(\psi|\partial H_1)^{-1}"'] & \partial H_2 \arrow[d, "id."] \\ \partial H_1 \arrow[r, "\phi\circ(\psi|\partial H_1)"'] & \partial H_2 \end{tikzcd}$$

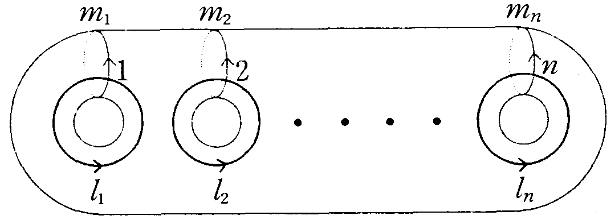
By the above Theorem, we can apply the handle sliding and handles combining to H-splitting to examine the changing of M^3.

Definition 5. Let (H_1, H_2, F) and (H_1', H_2', F') be H-splittings of M^3 with the same genus. If there exists a homeomorphism $f: M^3 \to M^3$ so that $f(F) = F'$, then (H_1, H_2, F) and (H_1', H_2', F') are called *equivalent*.

Definition 6. Suppose (H_1, H_2, F) is a genus $n(\geqq 1)$ H-splitting of M^3. Let $\{D_1, \cdots, D_n\}$,

$\{D_1', \cdots, D_n'\}$ be a complete system of meridian disks of H_1, H_2, respectively and $\{m\} = \{m_1, \cdots, m_n\} = \{\partial D_1, \cdots, \partial D_n\}$, $\{l\} = \{l_1, \cdots, l_n\} = \{\partial D_1', \cdots, \partial D_n'\}$. Then $(H_1 ; m, l)$ ($(H_2 ; l, m)$ resp.) is called *a genus n Heegaard diagram (H-diagram) associated with* (H_1, H_2, F). $\{m, l\}$($\{l, m\}$ resp.) are called *meridian-longitude systems of* $(H_1 ; m, l)$ ($(H_2 ; l, m)$ resp.).

By an ambient isotopy of H, a genus $n(\geqq 1)$ handlebody H is deformed such as shown in Fig. 2–4. This shows a genus n H-diagram $(H_1 ; m, l)$ of the 3-sphere. It is called *a canonical genus n H-diagram*.



$(H_1 ; m, l)$

Fig. 2–4

Let $(H_1 ; m_1, \cdots, m_n, l_1, \cdots, l_n)$ be a genus n H-diagram associated with (H_1, H_2, F) of M^3. We may assume that $(m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ consists at most of finite points (by an argument of general position).

Definition 7. The number of finite points of $\{m\} \cap \{l\} = (m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ is called *a cross point number with* $(H_1 ; m, l)$ or $(H_2 ; l, m)$.

3. Transformations of Heegaard diagrams

We begin with an obvious Proposition.

Proposition 3. *Let Fig. 2–5 be a part of H-diagram* $(U ; m, l)$. *The longitude* l_j *crosses the meridian* m_i, *turns back to* m_i *and crosses* m_i *again. Then, there exist a transformation of* $(U ; m, l)$ *so that a part of* l_j *deforms to the dotted line and it does not cross* m_i. *It does not change the H-genus but decreases the cross point number, as many as* 2.

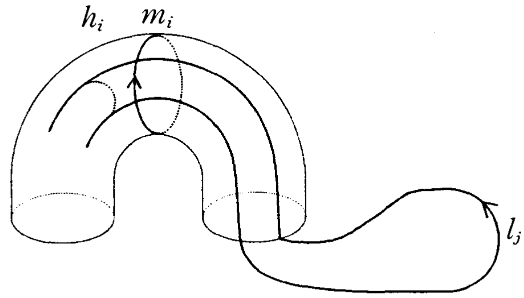


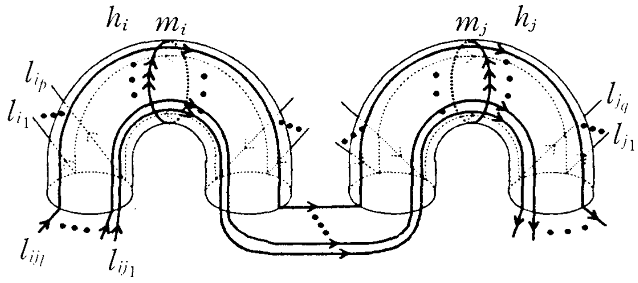
Fig. 2–5

Definition 8. The above transformation is called *a cancelling for the H-diagram*.
If the diagram like Fig. 2–5 appears, then we always do the above correction.

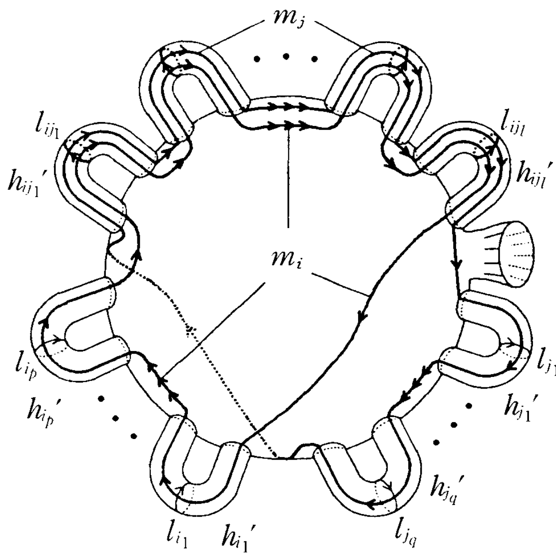
From now, we state Zieschang's result and give transformations by the band move for basic H-diagrams after that.

Theorem 4. *Let H be a genus* $n(\geqq 2)$ *handlebody. Then any two complete systems of meridian circles of* ∂H *transform each other under a finite sequence of band moves* (Zieschang [2]).

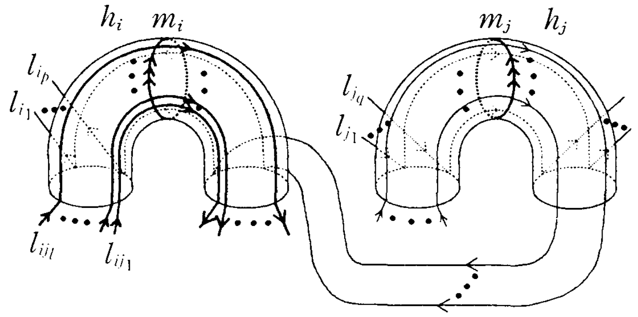
Let the following figure U1–A be a part of H-diagram (U ; m, l). The longitudes $\{l_{ij_1}, \cdots, l_{ij_l}\}$ ($l \geqq 0$) drawn heavily go around side by side on the two handles h_i and h_j. The longitudes $\{l_{i_1}, \cdots, l_{i_p}\}$, $\{l_{j_1}, \cdots, l_{j_q}\}$ go around on h_i, h_j, respectively. It shows the general case that longitudes run on handles h_i and h_j. In a special case that a character l on the lower right equals to 0, there are not longitudes that run on h_i and h_j. V1–A′ is the dual part of U1–A. The longitude m_i, m_j crosses the meridians $\{l_{i_1}, \cdots, l_{i_p}, l_{ij_1}, \cdots, l_{ij_l}\}$, $\{l_{j_1}, \cdots, l_{j_q}, l_{ij_1}, \cdots, l_{ij_l}\}$, respectively.



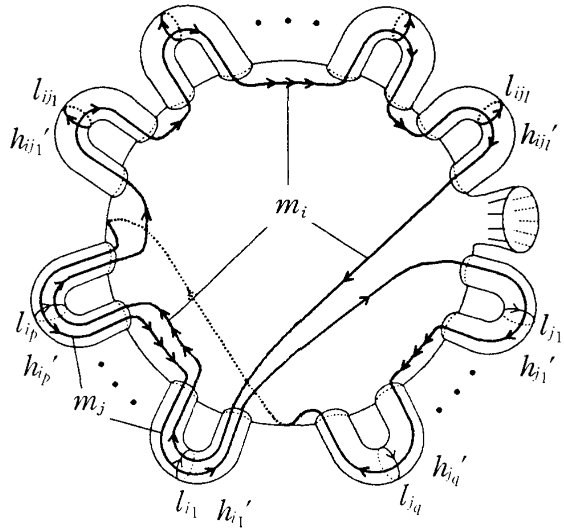
U1–A



V1–A′



U1–B

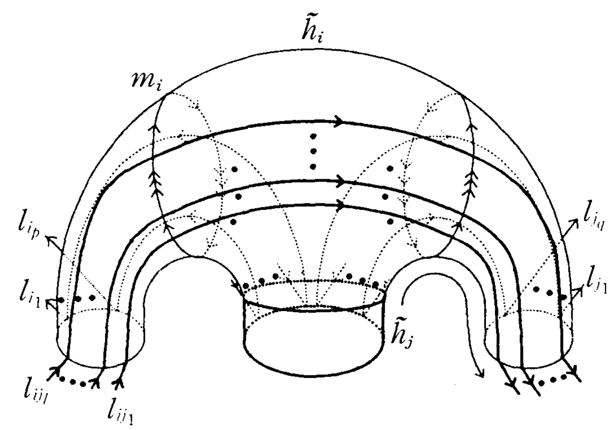


V1–B′

The transformation from U1−A into U1−B is obtained by the handle sliding of h_i about h_j along the directions of the longitudes $\{l_{ij_1}, \cdots, l_{ij_l}\}$ in $\partial(B_U{}^3+h_j)$. In U1−B, $\{l_{ij_1}, \cdots, l_{ij_l}\}$ go around on h_i (not on h_j), $\{l_{i_1}, \cdots, l_{i_p}\}$ go around on both the h_i and h_j, and $\{l_{j_1}, \cdots, l_{j_q}\}$ do not change the way of running. The dual transformation from V1−A′ into V1−B′ is obtained by a band move: each meridian l_{ij_k} is cut into two segments by the two longitudes m_i and m_j. Let the shorter segment be α. From $m_i \cup \alpha \cup m_j$, we may construct a band sum m_{ij} of m_i and m_j, and carry out a band move of m_j. By an ambient isotopy and reorienting m_j, V1−B′ is obtained. In V1−B′, m_i does not change the way of running, and here m_j comes to cross $\{l_{j_1}, \cdots, l_{j_q}, l_{i_p}, \cdots, l_{i_1}\}$.

In the case of $(l=0)$, if we can draw a band β which reaches to $m_j\times 0$ via $m_i\times 1$ as it does not intersect the longitudes, then we can handle sliding h_i about h_j along $\beta+\partial h_j$.

The above handle sliding is regarded as the band move of m_j: if we carry out the handles combining with h_i and h_j, then U1−C is obtained. The handle $\tilde{h}_j$ drawn heavily attaches under $\tilde{h}_i$. This means that handles combining gives a band move of m_j. Next by handle sliding of $\tilde{h}_j$ about $\tilde{h}_i$ along the direction of the line, U1−B is btained.



U1−C

In like manners, a handle sliding of h_j about h_i and a band move of m_i are obtained. Hence we have;

Theorem 5. *The transformation from U1−A(V1−A′ resp.) into U1−B (V1−B′ resp.) is carry out by a band move of m_j. It does not change the H-genus but changes the cross point number as many as $|l-p|$.*

In U1−A (V1−A′ resp.) we can carry out band moves for two meridians and two longitudes in Theorem 5. And band moves to transform H-diagrams are only these types.

Applying the Theorem 4 to both the $(U\,;m,\,l)$ and $(V\,;l,\,m)$ in Theorem 5, we have;

Theorem 6. *If $(U\,;m,\,l)\cup(V\,;l,\,m)$ and $(U\,;m',\,l')\cup(V\,;l',\,m')$ are two sets of the genus $n(\geqq 2)$ H-diagrams associated with $(U,\,V,\,F)$, then $(U\,;m,\,l)$ $((V\,;l,\,m)$ resp.) is transformed into $(U\,;m',\,l')$ $((V\,;l',\,m')$ resp.) under a finite sequence of band moves for two meridians and two longitudes.*

There is an important result about the equivalent of H-splittings for the 3-sphere.

Theorem 7. *H-splittings of the same genus of the 3-sphere are equivalent* (Waldhausen [3]).

The above Theorem means that it is made as it chooses meridian-longitude systems $\{m_1, \cdots, m_n\}$, $\{l_1, \cdots, l_n\}$ suitably in H-splitting of the 3-sphere, which satisfy the conditions of $m_i \cap l_j = \{\text{a point}\}$ $(i = j)$ and $m_i \cap l_j = \phi$ $(i \neq j)$.

From the Theorems 6 and 7, we have ;

Theorem 8. *Any genus $n(\geqq 2)$ H-diagram of the 3-sphere is transformed into the canonical one under a finite sequence of band moves for two meridians and two longitudes.*

It is not easy to transform H-diagrams. In [**5**, **6**], we obtain the methods of transformations of Heegaard cut diagrams (H-cut-diagrams) corresponding to those of H-diagrams. They are the ones which have applied DS-deformations (Ikeda, Yamashita, Yokoyama ; [**10**]) to H-cut-diagram for DS-diagram (Ikeda, Inoue ; [**7**], Ishii [8]). We see that D_n-deformation corresponds to the band move ([**6**]). In this way, DS-diagram and H-cut-diagram are closely related (Yamashita [**9**]).

4. Transformations of the fundamental groups

To state our result precisely, we prepare algebra calculations for groups.

Definition 9. Let $\langle a_1, \cdots, a_n \mid r_1 = 1, \cdots, r_m = 1 \rangle$ denotes a presentation of a finitely generated group, where $a_1, \cdots, a_n$ are generators and relator r_i is a word in the a_i^ε's $(\varepsilon = \pm 1)$. We underline to the letters which are operated.

Replacements letters ; if there are relations $\underline{a_i^\varepsilon a_j^\varepsilon} w_k = 1$ $(k = 1, \cdots, \alpha)$, then replace the generator a_i, letters $a_i^\varepsilon a_j^\varepsilon$ by a new letter $\tilde{a}_i$ (this becomes a new generator).

Substitution ; if there are two relations $w_1 \underline{a_{i_1}{}^\varepsilon \cdots a_{i_\alpha}{}^\varepsilon} = 1$ and $w_2 \underline{a_{i_1}{}^\varepsilon \cdots a_{i_\alpha}{}^\varepsilon} = 1$, where a_{i_k} $(k = 1, \cdots, \alpha)$ is a generator and $a_{i_1}{}^\varepsilon \cdots a_{i_\alpha}{}^\varepsilon$ is a common word, then substitute $a_{i_1}{}^\varepsilon \cdots a_{i_\alpha}{}^\varepsilon = w_1{}^{-1}$ for $w_2 a_{i_1}{}^\varepsilon \cdots a_{i_\alpha}{}^\varepsilon = 1$.

Each above algebra calculation preserves isomorphism of a group.

Let (U, V, F) be a genus $n(\geqq 1)$ H-splitting of M^3 and $(U ; m, l)$ a H-diagram of (U, V, F). $\{m\} = \{m_1, \cdots, m_n\}$ and $\{l\} = \{l_1, \cdots, l_n\}$ are meridian-longitude systems. Let each m_i, l_i be oriented. By applying the van Kampen's Theorem to $U \cup V$, we may obtain a well-known presentation of a fundamental group $\pi_1(M^3)$;

$$\pi_1(M^3) = \langle m_1, \cdots, m_n \mid \hat{l}_1 = 1, \cdots, \hat{l}_n = 1 \rangle \quad (1).$$

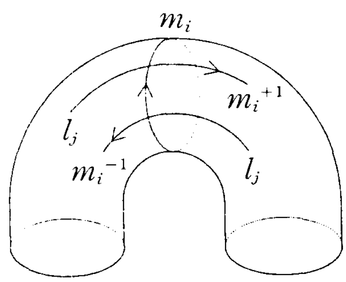


Fig. 4

We read that $m_1, \cdots, m_n$ are regarded as the generators of the meridians $m_1, \cdots, m_n$ and the relator $\hat{l}_j$ is a word in the $m_i{}^{\pm 1}$'s obtained by running once around the l_j, i.e., while we take a turn round l_j according to the orientation of l_j, we read the label m_i continuously as m_i^{+1} (m_i^{-1} resp.) if l_j crosses m_i from the left side (the right side resp.) to the right side (the left side resp.) of m_i. See Fig. 4. In the relator $\hat{l}_j$, we may start reading from any m_i in $\hat{l}_j$ because the word $\hat{l}_j$ becomes a cyclic word by joining both ends of $\hat{l}_j$ and preserving the sequential order of letters in $\hat{l}_j$. Hence $\hat{l}_j$ is uniquely defined up to cyclic permutations and inversions. A dual presentation from $(V\,;\,l,\,m)$ of $(U,\,V,\,F)$ is also defined in an analogous manner, and is denoted as

$$\pi_1(M^3) = \langle l_1,\,\cdots,\,l_n \,|\, \hat{m}_1 = 1,\,\cdots,\,\hat{m}_n = 1\rangle \quad (1').$$

Group (1) is isomorphic to (1′) but the presentation (1) is generally different from (1′) because meridians and longitudes are switched in $(U\,;\,m,\,l)$ and $(V\,;\,l,\,m)$. Hence the forms of relators in (1) and (1′) are different generally.

Let a presentation of the fundamental group derived from U1–A, U1–B of $(U\,;\,m,\,l)$ be (1A), (1B), respectively.

$$\left\langle \begin{array}{l|l} m_i,\, m_j & \underline{m_i m_j} w_{ij_k} = 1\cdots(l_{ij_k})\ (k = 1,\,\cdots,\,l) \\ m_k & m_i^{-1} w_{i_k} = 1\cdots(l_{i_k})\ (k = 1,\,\cdots,\,p) \\ (k \neq i,\,j) & m_j w_{j_k} = 1\cdots(l_{j_k})\ (k = 1,\,\cdots,\,q) \\ & r_\alpha = 1 \text{ (relations other than} \\ & \qquad \text{the above)} \end{array} \right\rangle \quad \text{(1A)}$$

$$\left\langle \begin{array}{l|l} m_i,\, m_j & m_i w_{ij_k} = 1\cdots(l_{ij_k})\ (k = 1,\,\cdots,\,l) \\ m_k & m_j m_i^{-1} w_{i_k} = 1\cdots(l_{i_k})\ (k = 1,\,\cdots,\,p) \\ (k \neq i,\,j) & m_j w_{j_k} = 1\cdots(l_{j_k})\ (k = 1,\,\cdots,\,q) \\ & r_\alpha = 1 \text{ (relations other than} \\ & \qquad \text{the above)} \end{array} \right\rangle \quad \text{(1B)}$$

Note that the relations (l_{ij_k}) $(k = 1,\,\cdots,\,l)$ in (1A) and (1B), too, do not exist if the longitudes l_{ij_k} $(k = 1,\,\cdots,\,l)$ do not exist.

operations ; in (1A), replace the generator m_i, letters $m_i m_j$ in (l_{ij_k}) by a new letter $\widetilde{m}_i$ (a new generator), we get a presentation that isomorphic to (1B).

Let a presentation of the fundamental group derived from V1–A′, V1–B′ of $(V\,;\,l,\,m)$ be (1A′), (1B′), respectively.

$$\left\langle \begin{array}{l|l} l_{i_1},\,\cdots,\,l_{i_p} & l_{i_1}^{-1}\cdots l_{i_p}^{-1}\underline{l_{ij_1}\cdots l_{ij_l}} = 1\cdots(m_i) \\ l_{j_1},\,\cdots,\,l_{j_q} & l_{j_1}\cdots l_{j_q}\underline{l_{ij_1}\cdots l_{ij_l}} = 1\cdots(m_j) \\ l_{ij_1},\,\cdots,\,l_{ij_l} & r_\alpha' = 1 \text{ (relations other than} \\ l_k\ (k \neq i,\,j,\,ij) & \qquad \text{the above)} \end{array} \right\rangle \quad \text{(1A}')$$

$$\left\langle \begin{array}{l|l} l_{i_1}, \cdots, l_{i_p} & l_{i_1}^{-1} \cdots l_{i_p}^{-1} l_{ij_1} \cdots l_{ij_l} = 1 \cdots (m_i) \\ l_{j_1}, \cdots, l_{j_q} & l_{j_1} \cdots l_{j_q} l_{i_p} \cdots l_{i_1} = 1 \cdots (m_j) \\ l_{ij_1}, \cdots, l_{ij_l} & r_\alpha' = 1 \text{ (relations other than} \\ l_k \ (k \neq i, j, ij) & \qquad \text{the above)} \end{array} \right\rangle \quad (1B')$$

Operation ; in (1A′), by substituting $l_{ij_1} \cdots l_{ij_l} = l_{i_p} \cdots l_{i_1}$ derived from (m_i) for $l_{ij_1} \cdots l_{ij_l}$ in (m_j), we get (1B′).

In like manner, transformations of the fundamental groups corresponding to those of a handle sliding of h_j about h_i of $(U ; m, l)$ and a band move of m_i of $(V ; l, m)$ are obtained.

Hence by gathering the Theorem 6 and considering the above, we have ;

Theorem 9. *Let $(H ; a, b)$ and $(H ; a', b')$ are the genus $n(\geqq 2)$ H-diagrams associated with a H-splitting of M^3. Then transformations from $(H ; a, b)$ into $(H ; a', b')$ by a finite sequence of band moves are in* 1-1 *correspondence with those of the fundamental group derived from* $(H ; a, b)$ by the replacements and substitutions.

Moreover, from the Theorems 8 and 9 we have ;

Theorem 10. *Any H-diagram of genus $n(\geqq 2)$ of the* 3*-sphere and the fundamental group derived from that are reduced to the canonical one and the trivial group by a finite sequence of band moves and corresponding the replacements and substitutions.*

We have a lot of examples for the 3-sphere S^3. Especially, they are ones about waves. The Whitehead [**11**]-Volodin-Kuznetsov-Fomenko [**12**] conjecture shows that "all H-diagrams of S^3 other than the canonical one have waves without fail." This is an algorithm for recognizing S^3 in 3-manifold. In [**1**], Birman describes that "nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counter example." In 1980, Homma-Ochiai-Takahashi [**14**] success in the above conjecture if H-genus = 2. But Viro [**13**], Morikawa [**15**], Ochiai [**17**] and the author [**6**] construct counter examples if H-genus $\geqq$ 3. We can realize the ones of the persons above as examples of Theorem 10.

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